Int. J. Solids Structures Vol. 29, No. 16, pp. 2077–2090, 1992
Printed in Great Britain.

0020-7683/92 $5.00 + .00

THERMOELASTIC INTERFACE CRACK PROBLEMS IN DISSIMILAR ANISOTROPIC MEDIA

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(*Received* 12 *February* 1991 ; *in revised form* 30 *November* 1991)

Abstract—By applying the extended version of Stroh's formalism and a special technique of analytical continuation, a general solution for the thermoelastic collinear interface cracks between dissimilar anisotropic media has been obtained in this paper. Special notification and examples are given for the application of the final results to the whole field solutions of the temperature, heat flux, displacements and stresses. The explicit expressions for the interface stresses and crack opening displacements are also provided. Based upon this general solution, two special examples of single crack problems are solved explicitly. One is homogeneous media, the other is anisotropic bimaterials. The former is studied for the purpose of presenting the detailed calculation, verifying the general solution and discussing the validity of the solution. A closed form solution is obtained for the latter case subjected to a uniform heat flux and loading.

1. INTRODUCTION

One of the most frequently encountered problems in composite laminates is interface cracking, sometimes also known as delamination. Delaminations in layered composite materials may occur due to a variety of reasons, such as low energy impact, manufacturing defects or high stress concentrations at geometry or material discontinuities (Kardomateas and Schmueser, 1988). Due to recent aerospace and commercial applications, laminated composites are experiencing an increased utilization in high temperature environments.

A quantitative assessment of the effect of realistic delaminations on the strength and lifetime of a laminate is difficult. Consequently, analytical efforts to date have only attempted to quantify the effect of idealized delaminations. Williams (1959) discovered the so-called oscillatory near tip behavior for an interface crack between two isotropic materials. Since then, many authors have discussed the interface crack problems such as England (1965), Erdogan (1965), Rice (1988) and Suo (1989) for isotropic media; and Gotoh (1967), Clements (1971), Willis (1971), Ting (1986), Bassani and Qu (1989), Qu and Bassani (1989), Suo (1990a), Wu (1990), Gao *et al.* (1992) and Hwu (1991) for anisotropic media. By a skill of orthotropy rescaling (Suo, 1990b) which reduces plane elasticity problems for orthotropic materials to equivalent problems for materials with cubic symmetry, certain interface crack solutions for orthotropic materials may also be obtained directly from isotropic solutions. In spite of the vast resources for the interface crack problems, very few published analytical studies are available for the thermoelastic interface crack problems. The steady state thermoelastic problems of interface cracks between dissimilar isotropic media have been studied by Erdogan (1965), Barber and Comninou (1982, 1983), Martin-Moran *et al.* (1983) and Sumi and Ueda (1990). As for the cracks between dissimilar anisotropic media, solutions were presented by Clements (1983) through the use of the complex variable method. Due to the lack of identities among the thermoelastic constants developed in later years (Ting, 1988; Hwu, 1990), the solutions provided by Clements (1983) are complicated. Moreover, without further notification those solutions can only be applied to interface, not the full field domain.

In this paper, we consider an arbitrary number of collinear cracks lying along the interface subjected to an arbitrary and self-equilibrated loading and heat flux on the upper and lower surfaces of the cracks. The materials are assumed to be perfectly bonded at all points except those lying in the region of cracks. To solve this problem, an extended version (Hwu, 1990) of Stroh's formalism (Stroh, 1958) for plane anisotropic thermoelasticity is applied. A special technique of analytical continuation (Muskhelishvili, 1954) which is similar to the one proposed by Suo (1990a), is also developed to consider the misfit of material constants along the interface. In Hwu's (1990) paper, the cracks are embedded in the homogeneous anisotropic media. While in Suo's (1990a) paper, no thermal effect is considered for the cracks lying along the interface of two dissimilar anisotropic media. The combination of the extended version of Stroh's formalism with the method of analytical continuation leads the problems of thermoelastic interface cracks to a Hilbert problem of vector form. A derivation of the general solution to the vector form Hilbert problem is provided in this paper. An explicit closed form solution for the thermoelastic interface crack problems is therefore obtained, which is expressed in complex matrix notation.

Special notification and examples are given for the application of the final results to the whole field solutions of temperature, heat flux, displacements and stresses. The explicit expressions for the interface stresses and crack opening displacements are also provided. The solutions are valid only when the assumption of fully open crack is not violated. If this is violated, a partial contact crack may be assumed, which is similar to the problems discussed by Martin-Moran *et al.* (1983) and Barber and Comninou (1983). Ting (1986) showed that if the negative imaginary part **W** of the bimaterial matrix **M*** (the definition can be found in this paper) is identical to zero, there will be no interpenetration problem for interface cracks. However, it has been argued (Rice, 1988) that the solutions can still be used to characterize the interface fracture process since the contact zone size is found to be extremely small for a broad range of bimaterial and loading configurations of practical importance. Hence, in this paper no detailed discussion about the validity of fully open crack assumption is provided. The main concern is devoted to the derivation of the general solution to the problems of thermoelastic interface collinear cracks without any restriction to the material properties of anisotropic media.

Two typical examples are solved explicitly. The simplest case when the two media are composed of the same material is studied for the purpose of presenting the detailed calculation and verifying the general solutions. A closed form solution is obtained for the case of bimaterial interface cracks subjected to uniform heat flux and loading.

2. PLANE ANISOTROPIC THERMOELASTICITY

2.1. *General solutions*

Based upon Stroh's formalism (Stroh, 1958) in anisotropic elasticity, a simple and compact version of general solutions for the uncoupled steady-state plane anisotropic thermoelasticity has been presented by Hwu (1990) as

$$T = 2\,\mathrm{Re}\,\{g'(z_t)\},\quad h_i = -2\,\mathrm{Re}\,\{(k_{i1}+\tau k_{i2})g''(z_t)\},\quad \mathbf{u} = 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{3}\mathbf{a}_\alpha f_\alpha(z_\alpha)+\mathbf{c}g(z_t)\right\},$$

$$\boldsymbol{\phi} = 2\,\mathrm{Re}\left\{\sum_{\alpha=1}^{3}\mathbf{b}_\alpha f_\alpha(z_\alpha)+\mathbf{d}g(z_t)\right\},\quad \sigma_{i1} = -\phi_{i,2},\quad \sigma_{i2} = \phi_{i,1}, \tag{1a}$$

where

$$z_\alpha = x_1+p_\alpha x_2,\quad z_t = x_1+\tau x_2. \tag{1b}$$

(x_1, x_2) is a fixed rectangular coordinate system. Re stands for the real part of a complex number. The prime (′) denotes differentiation with respect to its argument. A comma stands for differentiation. $T, h_i, \mathbf{u}, \boldsymbol{\phi}$ and σ_{ij} represent, respectively, the temperature, heat flux, displacements, stress functions and stresses. k_{ij} are the heat conduction coefficients. $f_\alpha(z_\alpha)$, $\alpha = 1, 2, 3$ and $g(z_t)$ are arbitrary functions with complex arguments z_α and z_t, respectively. p_α and $(\mathbf{a}_\alpha, \mathbf{b}_\alpha)$ are the elasticity eigenvalues with positive imaginary parts and the associated eigenvectors of

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{2a}$$

where

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 \\ \mathbf{N}_3 & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{\xi} = \begin{Bmatrix} \mathbf{a} \\ \mathbf{b} \end{Bmatrix}, \tag{2b}$$

$$\mathbf{N}_1 = \mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}}, \quad \mathbf{N}_2 = \mathbf{T}^{-1} = \mathbf{N}_2^{\mathrm{T}}, \quad \mathbf{N}_3 = \mathbf{R}\mathbf{T}^{-1}\mathbf{R}^{\mathrm{T}} - \mathbf{Q} = \mathbf{N}_3^{\mathrm{T}}, \tag{2c}$$

and

$$Q_{ik} = C_{i1k1}, \quad R_{ik} = C_{i1k2}, \quad T_{ik} = C_{i2k2}. \tag{2d}$$

The superscript T denotes the transpose. C_{ijkl} are the elastic constants which are assumed to be fully symmetric and positive definite so that the strain energy is positive. τ and $(\mathbf{c}, \mathbf{d})$ are the heat eigenvalue with positive imaginary part and the associated generalized eigenvectors of

$$k_{22}\tau^2 + 2k_{12}\tau + k_{11} = 0, \quad \mathbf{N}\boldsymbol{\eta} = \tau\boldsymbol{\eta} + \boldsymbol{\gamma}, \tag{3a}$$

where

$$\boldsymbol{\gamma} = -\begin{bmatrix} \mathbf{0} & \mathbf{N}_2 \\ \mathbf{I} & \mathbf{N}_1^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\beta}_1 \\ \boldsymbol{\beta}_2 \end{Bmatrix}, \quad \boldsymbol{\eta} = \begin{Bmatrix} \mathbf{c} \\ \mathbf{d} \end{Bmatrix}, \tag{3b}$$

and

$$(\boldsymbol{\beta}_1)_i = \beta_{i1}, \quad (\boldsymbol{\beta}_2)_i = \beta_{i2}. \tag{3c}$$

β_{ij} are the thermal moduli which are assumed to be symmetric. In eqn (3a), the symmetry assumption of k_{ij} has been employed. Note that the general solution given in (1) is obtained under the assumption that the heat eigenvalue and the elasticity eigenvalues are distinct. If they are repeated, a small perturbation of the material constants can be employed to avoid the degenerate problem. Otherwise, a modified solution should be applied (Wu, 1984). However, if the final solutions do not contain the eigenvectors $\mathbf{a}_\alpha$, $\mathbf{b}_\alpha$ and $\mathbf{c}$, $\mathbf{d}$, the problems of repeated eigenvalues can then be avoided, which can usually be achieved through the use of the identities given in the next subsection.

As shown by Suo (1990a) the analyticity of a function is not affected by the arguments z_α, $\alpha = 1, 2, 3$, or z_t. Another solution form appropriate for the method of analytic continuation (Muskhelishvili, 1954) is written as

$$T = g'(z) + \overline{g'(z)}, \quad h_i = -(k_{i1} + \tau k_{i2})g''(z) - (k_{i1} + \bar{\tau} k_{i2})\overline{g''(z)},$$
$$\mathbf{u} = \mathbf{A}\mathbf{f}(z) + \mathbf{c}g(z) + \bar{\mathbf{A}}\overline{\mathbf{f}(z)} + \bar{\mathbf{c}}\overline{g(z)}, \quad \boldsymbol{\phi} = \mathbf{B}\mathbf{f}(z) + \mathbf{d}g(z) + \bar{\mathbf{B}}\overline{\mathbf{f}(z)} + \bar{\mathbf{d}}\overline{g(z)}, \tag{4a}$$

where

$$\mathbf{A} = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \mathbf{a}_3], \qquad \mathbf{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \mathbf{b}_3], \qquad \mathbf{f}(z) = [f_1(z) \quad f_2(z) \quad f_3(z)]^{\mathrm{T}}, \tag{4b}$$

and the overbar represents the conjugate of a complex number. Note that the arguments of $g(z)$ and each component function of $\mathbf{f}(z)$ are written as $z = x_1 + px_2$ without referring to the associated eigenvalues τ or p_α. Once the solutions of $g(z)$ and $\mathbf{f}(z)$ are obtained for a given boundary value problem, a replacement of z_t, z_1, z_2 or z_3 should be made for each function to calculate field quantities from (1) or (4).

2.2. *Identities*

Due to the orthogonality relation among the eigenvectors $\boldsymbol{\xi}_\alpha$ derived by Stroh (1958), three real matrices have been introduced as

$$\mathbf{S} = \mathrm{i}(2\mathbf{A}\mathbf{B}^{\mathrm{T}} - \mathbf{I}), \quad \mathbf{H} = 2\mathrm{i}\mathbf{A}\mathbf{A}^{\mathrm{T}}, \quad \mathbf{L} = -2\mathrm{i}\mathbf{B}\mathbf{B}^{\mathrm{T}}, \tag{5}$$

in which $\mathrm{i} = \sqrt{-1}$ is a pure imaginary number and $\mathbf{I}$ is the unit matrix. $\mathbf{H}$ and $\mathbf{L}$ are symmetric and positive definite and $\mathbf{SH}$, $\mathbf{LS}$, $\mathbf{H}^{-1}\mathbf{S}$, $\mathbf{SL}^{-1}$ are anti-symmetric. From the above relations, the impedance matrix **M** (Ingebrigtsen and Tonning, 1969) which has been used widely for the interface crack problems can be shown to be

$$\mathbf{M} = -i\mathbf{B}\mathbf{A}^{-1} = \mathbf{H}^{-1}(\mathbf{I}+i\mathbf{S}) = (\mathbf{I}-i\mathbf{S}^T)\mathbf{H}^{-1},$$

$$\mathbf{M}^{-1} = i\mathbf{A}\mathbf{B}^{-1} = \mathbf{L}^{-1}(\mathbf{I}+i\mathbf{S}^T) = (\mathbf{I}-i\mathbf{S})\mathbf{L}^{-1}. \tag{6}$$

The third equalities in (6) come from the fact that $\mathbf{H}^{-1}\mathbf{S}$ and $\mathbf{S}\mathbf{L}^{-1}$ are anti-symmetric. Hence **M** is a Hermitian matrix. Another identity related to the thermoelastic properties is (Hwu, 1990)

$$\mathbf{S}\mathbf{c}+\mathbf{H}\mathbf{d} = i\mathbf{c}+\tilde{\gamma}_1^*, \quad -\mathbf{L}\mathbf{c}+\mathbf{S}^T\mathbf{d} = i\mathbf{d}+\tilde{\gamma}_2^*, \tag{7a}$$

where

$$\tilde{\gamma}_1^* = \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta+\tau\sin\theta)^{-1}\gamma_1(\theta)\,\mathrm{d}\theta, \quad \gamma_1(\theta) = -\mathbf{N}_2(\theta)\boldsymbol{\beta}\mathbf{m}(\theta),$$

$$\tilde{\gamma}_2^* = \frac{1}{2\pi}\int_0^{2\pi}(\cos\theta+\tau\sin\theta)^{-1}\gamma_2(\theta)\,\mathrm{d}\theta, \quad \gamma_2(\theta) = -\boldsymbol{\beta}\mathbf{n}(\theta)-\mathbf{N}_1^T(\theta)\boldsymbol{\beta}\mathbf{m}(\theta), \tag{7b}$$

and

$$\mathbf{n}(\theta) = (\cos\theta\ \sin\theta\ 0)^T, \quad \mathbf{m}(\theta) = (-\sin\theta\ \cos\theta\ 0)^T, \quad \boldsymbol{\beta} = [\boldsymbol{\beta}_1\ \boldsymbol{\beta}_2\ \boldsymbol{\beta}_3]. \tag{7c}$$

$\mathbf{N}_i(\theta)$, $i = 1, 2, 3$ are the generalized forms of $\mathbf{N}_i$ (Hwu, 1990).

3. THERMOELASTIC COLLINEAR INTERFACE CRACKS

Consider an arbitrary number of collinear cracks lying along the interface of two dissimilar anisotropic materials. The materials are assumed to be perfectly bonded at all points of the interface $x_2 = 0$ except those lying in the region of cracks L (see Fig. 1), which are defined by the intervals

$$a_j \leqslant x_1 \leqslant b_j, \quad j = 1, 2, \ldots, n,$$

with

$$-\infty < a_1 < b_1 < a_2 < b_2 < \cdots < a_n < b_n < \infty.$$

On the upper and lower surfaces of the cracks, an arbitrary and self-equilibrated loading and heat flux are specified. From Section 2, we know that the solution to an individual

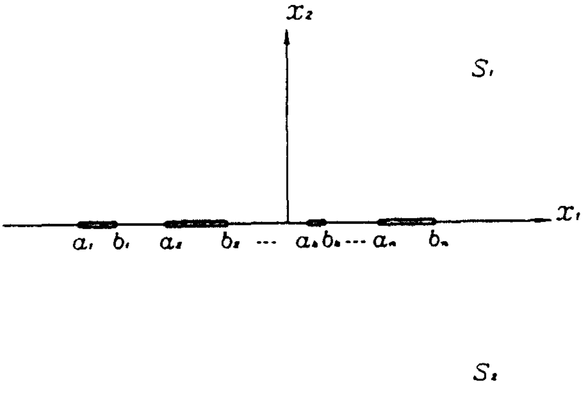

Fig. 1. Collinear interface cracks.

problem in two-dimensional anisotropic thermoelasticity can be reduced to finding the complex functions $\mathbf{f}$ and g, which should satisfy the boundary conditions of that problem. In the case of two different materials, however, the elastic properties are discontinuous across the bonded line, and a complete solution to the problem requires the knowledge of two complex function vectors $\mathbf{f}_1$, $\mathbf{f}_2$ and two complex scalar functions g_1, g_2. Here, the subscripts 1 and 2 are used to denote the quantities pertaining to the materials 1 and 2 which are located on $x_2 > 0$ (S_1) and $x_2 < 0$ (S_2), respectively. The functions $\mathbf{f}_1, g_1$ and $\mathbf{f}_2, g_2$ are holomorphic in the regions S_1 and S_2, respectively. They are sought to satisfy the continuity of displacement, traction, temperature and heat flux across the bonded portion of the interface, as well as the prescribed traction and heat flux conditions on the crack portion, i.e.

$$\begin{gathered}\mathbf{u}_1 = \mathbf{u}_2, \quad \boldsymbol{\phi}_1 = \boldsymbol{\phi}_2, \quad T_1 = T_2, \quad (h_2)_1 = (h_2)_2, \quad x_1 \notin L, \\ \boldsymbol{\phi}'_1 = \boldsymbol{\phi}'_2 = \hat{\mathbf{t}}, \quad (h_2)_1 = (h_2)_2 = \hat{h}, \quad x_1 \in L.\end{gathered} \tag{8}$$

The equality of traction continuity comes from the relation $\partial\boldsymbol{\phi}/\partial s = \mathbf{t}$ where $\mathbf{t}$ is the surface traction on a curve boundary and s is the arc length measured along the curved boundary. $\hat{\mathbf{t}}$ and $\hat{h}$ are the prescribed traction and heat flux applied on the upper and lower surfaces of the cracks. When the points along the crack surfaces are considered, integration of $\boldsymbol{\phi}'_1 = \boldsymbol{\phi}'_2$ provides $\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2$ since the integration constants can be neglected, which correspond to rigid body motion. Combining this result with the continuity requirements of traction and heat flux along the bonded portion, we have

$$\boldsymbol{\phi}_1 = \boldsymbol{\phi}_2, \quad (h_2)_1 = (h_2)_2, \quad \text{along the entire interface.} \tag{9}$$

By introducing a real constant $k = k_{22}(\tau - \bar{\tau})/2i$, the expression of the heat flux component h_2 given in $(4a)_2$ can be simplified as

$$h_2 = -ikg''(z) + \overline{ikg''(z)}. \tag{10}$$

Using eqn (10), the heat flux continuity condition $(9)_2$ leads to

$$-ik_1g''_1(x_1^+) - ik_2\overline{g''_2(x_1^-)} = -ik_2g''_2(x_1^-) - ik_1\overline{g''_1(x_1^+)}, \tag{11}$$

where $x_1^{\pm}$ denote, respectively, the points on the upper and lower surfaces of the cracks. One of the important properties of holomorphic functions used in the method of analytical continuation is that if $\mathbf{f}(z)$ is holomorphic in S_1 (or S_2), then $\overline{\mathbf{f}(\bar{z})}$ is holomorphic in S_2 (or S_1). From this property and eqn (11), we may introduce a function which is holomorphic in the entire domain including the interface, i.e.

$$g^*(z) = \begin{cases} -ik_1g''_1(z) - ik_2\overline{g''_2(\bar{z})}, & z \in S_1, \\ -ik_2g''_2(z) - ik_1\overline{g''_1(\bar{z})}, & z \in S_2. \end{cases} \tag{12}$$

Since $g^*(z)$ is now holomorphic and single-valued in the whole plane including the point at infinity, by Liouville's Theorem we have $g^*(z) \equiv$ constant. If the heat flux tends to zero when $|z| \to \infty$, $g^*(z)$ is then identically zero, i.e.

$$g^*(z) \equiv 0. \tag{13}$$

If the temperature field also tends to zero as $|z| \to \infty$, and the terms corresponding to the rigid body motion are neglected, combining (12) and (13) we have

$$\overline{g_2}(z) = -\frac{k_1}{k_2}g_1(z), \quad z \in S_1, \qquad \overline{g_1}(z) = -\frac{k_2}{k_1}g_2(z), \quad z \in S_2. \tag{14}$$

Similarly, by applying $(4a)_3$, the traction continuity $(9)_1$ along the entire interface and the result of (14), the use of the analytical continuation method leads to

$$\begin{aligned} \bar{\mathbf{f}}_2(z) &= \bar{\mathbf{B}}_2^{-1}\left[\mathbf{B}_1\mathbf{f}_1(z) + \left(\mathbf{d}_1 + \frac{k_1}{k_2}\bar{\mathbf{d}}_2\right)g_1(z)\right], \quad z \in S_1, \\ \bar{\mathbf{f}}_1(z) &= \bar{B}_1^{-1}\left[\mathbf{B}_2\mathbf{f}_2(z) + \left(\mathbf{d}_2 + \frac{k_2}{k_1}\bar{\mathbf{d}}_1\right)g_2(z)\right], \quad z \in S_2. \end{aligned} \tag{15}$$

By employing $(4a)_{1,3}$, (14) and (15), the temperature and displacement continuity along the bonded portion of the interface now provide

$$\theta(x_1^+) = \theta(x_1^-), \quad \psi(x_1^+) = \psi(x_1^-), \quad x_1 \notin L, \tag{16a}$$

where

$$\theta(z) = \begin{cases} \left(1 + \dfrac{k_1}{k_2}\right)g_1(z), & z \in S_1, \\ \left(1 + \dfrac{k_2}{k_1}\right)g_2(z), & z \in S_2, \end{cases} \tag{16b}$$

$$\psi(z) = \begin{cases} \bar{\mathbf{B}}_1\mathbf{f}_1(z) + \dfrac{k_2}{k_1 + k_2}\mathbf{d}_1 + \mathbf{e}_1, & z \in S_1, \\ \mathbf{M}^{*-1}\bar{\mathbf{M}}^*\mathbf{B}_2\mathbf{f}_2(z) + \dfrac{k_1}{k_1 + k_2}\mathbf{d}_2 + \mathbf{e}_2, & z \in S_2. \end{cases} \tag{16c}$$

In the above, $\mathbf{M}^*$ is the bimaterial matrix defined as

$$\mathbf{M}^* = \mathbf{M}_1^{-1} + \bar{\mathbf{M}}_2^{-1} = \mathrm{i}(\mathbf{A}_1\mathbf{B}_1^{-1} - \bar{\mathbf{A}}_2\bar{\mathbf{B}}_2^{-1}) = -\mathrm{i}(\mathbf{W} + \mathrm{i}\mathbf{D}), \tag{17a}$$

where

$$\mathbf{W} = \mathbf{S}_1\mathbf{L}_1^{-1} - \mathbf{S}_2\mathbf{L}_2^{-1}, \quad \mathbf{D} = \mathbf{L}_1^{-1} + \mathbf{L}_2^{-1}. \tag{17b}$$

The second and third equalities of (17a) come from the identities given in (6). The complex vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ are related to the heat eigenvectors **c** and **d**, and are defined as

$$\begin{aligned} \mathbf{e}_1 &= \frac{1}{k_1 + k_2}[\mathrm{i}\mathbf{M}^{*-1}(\mathbf{c}^* - \mathrm{i}\bar{\mathbf{M}}_2^{-1}\mathbf{d}^*) - k_2\mathbf{d}_1], \\ \mathbf{e}_2 &= \frac{1}{k_1 + k_2}[\mathrm{i}\bar{\mathbf{M}}^{*-1}(\bar{\mathbf{c}}^* - \mathrm{i}\bar{\mathbf{M}}_1^{-1}\bar{\mathbf{d}}^*) - k_1\mathbf{d}_2], \end{aligned} \tag{18a}$$

where

$$\mathbf{c}^* = k_2\mathbf{c}_1 + k_1\bar{\mathbf{c}}_2, \quad \mathbf{d}^* = k_2\mathbf{d}_1 + k_1\bar{\mathbf{d}}_2. \tag{18b}$$

Using the results of (14), (15) and (16b, c), the prescribed traction and heat flux conditions on the crack portion $(8b)_{1,2}$ lead to the following Hilbert problems (Muskhelishvili, 1954)

$$\theta''(x_1^+)+\theta''(x_1^-) = \frac{\mathrm{i}(k_1+k_2)}{k_1k_2}\hat{h}(x_1),$$

$$\boldsymbol{\psi}'(x_1^+)+\bar{\mathbf{M}}^{*-1}\mathbf{M}^*\boldsymbol{\psi}'(x_1^-) = \hat{\mathbf{t}}(x_1)+\theta'(x_1^+)\mathbf{e}_1+\theta'(x_1^-)\mathbf{e}_2, \quad x_1\in L. \tag{19}$$

The solutions to these Hilbert problems are (see Appendix)

$$\theta''(z) = \frac{k_1+k_2}{2\pi k_1k_2}\chi_0(z)\int_L \frac{\hat{h}(s)\,\mathrm{d}s}{\chi_0^+(s)(s-z)}+\chi_0(z)p_n(z) \tag{20a}$$

$$\boldsymbol{\psi}'(z) = \frac{1}{2\pi\mathrm{i}}\mathbf{X}_0(z)\int_L \frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}[\hat{\mathbf{t}}(s)+\theta'(s^+)\mathbf{e}_1+\theta'(s^-)\mathbf{e}_2]\,\mathrm{d}s+\mathbf{X}_0(z)\mathbf{p}_n(z) \tag{20b}$$

where $p_n(z)$ and $\mathbf{p}_n(z)$ are arbitrary polynomials with the degree not higher than n, and χ_0, $\mathbf{X}_0(z)$ are the basic Plemelj functions defined as,

$$\chi_0(z) = \prod_{j=1}^{n}(z-a_j)^{-1/2}(z-b_j)^{-1/2}, \quad \mathbf{X}_0(z) = \mathbf{\Lambda}\mathbf{\Gamma}(z), \tag{20c}$$

where

$$\mathbf{\Lambda} = [\boldsymbol{\lambda}_1 \quad \boldsymbol{\lambda}_2 \quad \boldsymbol{\lambda}_3], \qquad \mathbf{\Gamma}(z) = \left\langle\left\langle \prod_{j=1}^{n}(z-a_j)^{-(1+\delta_\alpha)}(z-b_j)^{\delta_\alpha}\right\rangle\right\rangle. \tag{20d}$$

The angular bracket $\langle\langle\ \rangle\rangle$ stands for the diagonal matrix, i.e.

$$\langle\langle f_\alpha\rangle\rangle = \mathrm{diag}[f_1 \quad f_2 \quad f_3]$$

which will be used throughout this paper, δ_α and $\boldsymbol{\lambda}_\alpha$, $\alpha = 1, 2, 3$, of (20d) are the eigenvalues and eigenvectors of

$$(\mathbf{M}^*+\mathrm{e}^{2\mathrm{i}\pi\delta}\bar{\mathbf{M}}^*)\boldsymbol{\lambda} = \mathbf{0}. \tag{20e}$$

The explicit solution for the eigenvalue δ has been given by Ting (1986) as

$$\delta_\alpha = -\tfrac{1}{2}+\mathrm{i}\varepsilon_\alpha, \quad \alpha = 1,2,3, \tag{20f}$$

where

$$\varepsilon_1 = \varepsilon = \frac{1}{2\pi}\ln\frac{1+\omega}{1-\omega}, \quad \varepsilon_2 = -\varepsilon, \quad \varepsilon_3 = 0, \quad \omega = [-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{W}\mathbf{D}^{-1})^2]^{1/2}, \tag{20g}$$

tr stands for the trace of the matrix. Note that the order of singularity δ_α is independent of the heat conduction coefficients k_{ij} and thermal moduli β_{ij}, and is the same as those of the isothermal interface crack problems. The order of singularity related to the heat flux is $-1/2$ as shown in (20c).

Once we get the solution of $\theta''(z)$ and $\boldsymbol{\psi}'(z)$ from (20), the complex functions $g_1(z)$, $g_2(z)$ and $\mathbf{f}_1(z)$, $\mathbf{f}_2(z)$ can be obtained from (16b, c) with the understanding that the subscript of z is dropped since the analytical continuation is not affected by different arguments z_t or z_α. After the operation of matrices, a replacement of z_t or z_1, z_2, z_3 should be made for each function, because the functions $g_k(z)$ and $\mathbf{f}_k(z)$ are required to have the form

$$g_k(z) = g_k(z_t), \quad \mathbf{f}_k(z) = [f_1(z_1) f_2(z_2) f_3(z_3)]^{\mathrm{T}}, \quad k = 1,2$$

which can be seen from the general solution given in (1). This calculation procedure will be applied throughout this paper. The whole field solution can then be found by using eqn (4). If one is interested in the stresses σ_{i2} along the interface and the crack opening displacements $\Delta\mathbf{u}$, the following results show that they have a simple relation with functions $\boldsymbol{\psi}(z)$ and $\theta(z)$. By applying $\sigma_{i2} = \phi_{i,1}$, $(4a)_4$, (14), (15) and (16b, c), the stresses σ_{i2} are calculated as

$$\begin{Bmatrix} \sigma_{12} \\ \sigma_{22} \\ \sigma_{32} \end{Bmatrix} = \boldsymbol{\phi}' = (\mathbf{I}+\bar{\mathbf{M}}^{*-1}\mathbf{M}^*)\boldsymbol{\psi}'(x_1) - \theta'(x_1)(\mathbf{e}_1+\mathbf{e}_2), \quad x_1 \notin L. \tag{21}$$

From $(4a)_3$, (14), (15) and (16b, c), the crack opening displacements $\Delta\mathbf{u}$ can also be calculated and simplified as

$$\Delta\mathbf{u} = \mathbf{u}(x_1, 0^+) - \mathbf{u}(x_1, 0^-), \quad = -\mathrm{i}\mathbf{M}^*[\boldsymbol{\psi}(x_1^+) - \boldsymbol{\psi}(x_1^-)], \quad x_1 \in L. \tag{22}$$

4. EXAMPLES

4.1. *Homogeneous media*

The simplest case of the interface cracks is when the two media are composed of the same materials. Our results for the thermoelastic interface crack problems should therefore be checked by this simplest case. Consider an infinite homogeneous anisotropic plate containing an insulated crack in which the heat is flowing uniformly in the direction of the positive x_2-axis. Due to the linear property, the principle of superposition can be used and the problem can be represented as the sum of a uniform heat flux in an uncracked solid and corrective problem which is described by

$$\hat{h}(x_1) = -h_0 = \text{constant}, \quad \hat{\mathbf{t}}(x_1) = \mathbf{0}, \quad n = 1, \quad a_1 = -a, \quad b_1 = a,$$
$$\mathbf{A}_1 = \mathbf{A}_2 = \mathbf{A}, \quad \mathbf{B}_1 = \mathbf{B}_2 = \mathbf{B}, \quad \mathbf{c}_1 = \mathbf{c}_2 = \mathbf{c}, \quad \mathbf{d}_1 = \mathbf{d}_2 = \mathbf{d}, \quad k_1 = k_2 = k. \tag{23}$$

To find the solution for this corrective problem, the line integral given in (20a) should be evaluated first. By residue theory, the integral around a closed contour C shown in Fig. 2 can be calculated as

$$\oint \frac{h_0\,\mathrm{d}s}{\chi_0^{+}(s)(s-z)} = 2\pi\mathrm{i} \sum_{k=1}^{n} r_k, \quad \chi_0(s) = \frac{1}{\sqrt{s^2-a^2}},$$

where r_k is the residue of the integrand at its singular points within C. The closed contour C is the union of L^+, C_0, L^-, L_1, C_∞, L_1', C_0'. The summation of the integrals along L_1 and L_1' vanishes since they have opposite directions and the integrand across this line is continuous. The integrals around the circles C_0 and C_0' can be proved to be zero when the

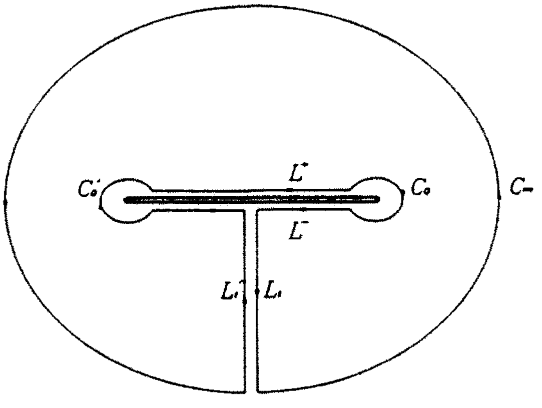

Fig. 2. Integration contour.

radii of the circles C_0 and C'_0 tend to zero. By replacing the contour of C_∞ by $Re^{i\psi}$ and letting $R \to \infty$, the integral around C_∞ is found to be

$$\int_{C_\infty} \frac{h_0 \, \mathrm{d}s}{\chi_0(s)(s-z)} = 2\pi h_0 \mathrm{i} z.$$

Knowing that $\chi_0(z)$ is the homogeneous solution of the Hilbert problem, i.e. $\chi_0^+ + \chi_0^- = 0$, we have

$$\int_{L^+ + L^-} \frac{h_0 \, \mathrm{d}s}{\chi_0(s)(s-z)} = 2 \int_L \frac{h_0 \, \mathrm{d}s}{\chi_0^+(s)(s-z)}.$$

The only pole which has made a contribution to the residues is at $s = z$, and the residue at that point is $h_0 \chi_0^{-1}(z)$. With the above description, we are now in a position to evaluate the line integral and the final simplified result is

$$\int_L \frac{h_0 \, \mathrm{d}s}{\chi_0^+(s)(s-z)} = -\pi h_0 \mathrm{i}(z - \sqrt{z^2 - a^2}). \tag{24a}$$

After evaluating the line integral, the arbitrary polynomial

$$p_1(z) = c_0 + c_1 z, \tag{24b}$$

could be determined by the infinity condition and the single-valuedness requirement. If $\theta''(z) \to 0$ as $|z| \to \infty$, we have

$$c_1 = 0. \tag{24c}$$

The requirement of the single-valuedness condition can be expressed by

$$\int_{-a}^{a} [\theta''(x_1^+) - \theta''(x_1^-)] \, \mathrm{d}x_1 = 0. \tag{24d}$$

Knowing that $\sqrt{z^2 - a^2} = \pm \mathrm{i}\sqrt{a^2 - x_1^2}$ for $|x_1| < a$ and $x_2 = \pm 0$, substitution of (24a–c) and (20a) into (24d) leads to

$$c_0 = 0. \tag{24e}$$

Combining eqns (24a–e), the final simplified result for $\theta''(z)$ is

$$\theta''(z) = -\frac{\mathrm{i}h_0}{k}\left(1 - \frac{z}{\sqrt{z^2 - a^2}}\right). \tag{25}$$

To find the solution for $\psi'(z)$, we first calculate the terms related to $\theta'(z)$. Integrating $\theta''(z)$ given in (25) with the assumption that the temperature field tends to zero as $|z| \to \infty$, and substituting the identities (6) and (7a) into (18) with the homogeneous condition $(23)_{3,4}$, we have

$$\theta'(s^+)\mathbf{e}_1 + \theta'(s^-)\mathbf{e}_2 = -\frac{h_0 s}{k} \operatorname{Re}\{\tilde{\gamma}_2^*\}.$$

The evaluation of the line integral and the determination of the arbitrary polynomial are similar to those described in eqns (24a–e). The result is

$$\psi'(z) = -\frac{h_0}{4k}\left[2z - \frac{2z^2-a^2}{\sqrt{z^2-a^2}}\right]\mathrm{Re}\,\{\tilde{\gamma}_2^*\}. \tag{26}$$

The solutions of the complex functions $g(z)$ and $\mathbf{f}(z)$ are obtained from (16b, c) with the aid of $(23)_{3,4}$ and $(7a)_2$ as

$$g(z) = \tfrac{1}{2}\theta(z), \quad \mathbf{f}(z) = \tfrac{1}{2}\mathbf{B}^{-1}\{2\psi(z) - \theta(z)[\mathbf{d} - \mathrm{i}\,\mathrm{Re}\,(\tilde{\gamma}_2^*)]\}. \tag{27}$$

Note again that the subscript of z is dropped before the multiplication of matrices and a replacement of z_t, z_α should be made for each component function after the matrix product. By this calculation procedure, the explicit expressions for the complex functions $g(z)$ and $\mathbf{f}(z)$ can be obtained from (25)–(27) as

$$g(z) = -\frac{\mathrm{i}h_0}{4k}\{z_t^2 - z_t\sqrt{z_t^2-a^2} + a^2\log(z_t + \sqrt{z_t^2-a^2})\},$$

$$\mathbf{f}(z) = \frac{\mathrm{i}h_0}{4k}\langle\langle z_\alpha^2 - z_\alpha\sqrt{z_\alpha^2-a^2}\rangle\rangle\mathbf{B}^{-1}\mathbf{d}$$

$$+ \frac{\mathrm{i}h_0a^2}{4k}\langle\langle \log(z_\alpha + \sqrt{z_\alpha - a^2})\rangle\rangle\mathbf{B}^{-1}[\mathbf{d} - \mathrm{i}\,\mathrm{Re}\,(\tilde{\gamma}_2^*)], \tag{28}$$

which can be proved to be identical to those presented in Hwu (1990) through the use of identities given in (5) and (7a). The whole field solutions for the temperature, heat flux, displacements and stresses can then be found by using eqn (4). The stresses σ_{i2} ahead of the crack tip along the x_1-axis are calculated by (21) with θ and ψ given in (25) and (26) as

$$\sigma_{i2} = \frac{h_0}{2k}\frac{a^2}{\sqrt{x_1^2-a^2}}\mathrm{Re}\,\{\tilde{\gamma}_2^*\}. \tag{29a}$$

Same as the isothermal problems, the above solution shows that the stresses are singular near the crack tip. With the usual definition, the stress intensity factors are given by

$$\begin{Bmatrix} K_{II} \\ K_I \\ K_{III} \end{Bmatrix} = \lim_{x_1 \to a}\sqrt{2\pi(x_1-a)}\,\sigma_{i2} = \frac{\sqrt{\pi}h_0}{2k}a^{3/2}\,\mathrm{Re}\,\{\tilde{\gamma}_2^*\}. \tag{29b}$$

Similarly, the crack opening displacements $\Delta\mathbf{u}$ are obtained from (22) and (26) as

$$\Delta\mathbf{u} = \mathbf{u}(x_1, 0^+) - \mathbf{u}(x_1, 0^-) = \frac{h_0}{k}x_1\sqrt{a^2-x_1^2}\,\mathbf{L}^{-1}\,\mathrm{Re}\,\{\tilde{\gamma}_2^*\}. \tag{29c}$$

The validity of this solution related to the assumption of a fully open crack has been discussed by Hwu (1990). By applying the virtual crack closure method (Irwin, 1957), the total strain energy release rate G can then be calculated as

$$G = \lim_{\Delta a \to 0}\frac{1}{2\Delta a}\int_0^{\Delta a} u_i(x-\Delta a)\sigma_{i2}(x)\,\mathrm{d}x = \frac{\pi h_0^2 a^3}{8k^2}\,\mathrm{Re}\,\{\tilde{\gamma}_2^*\}^{\mathrm{T}}\mathbf{L}^{-1}\,\mathrm{Re}\,\{\tilde{\gamma}_2^*\}. \tag{29d}$$

The solutions given in eqns (29a–d) are also exactly the same as those presented in Hwu (1990).

4.2. *Bimaterials*

Consider an interface crack located on $a_1 = -a$, $b_1 = a$, subjected to uniform heat flux $\hat{h} = -h_0$ and uniform loading $\hat{\mathbf{t}} = -\mathbf{t}_0$. To find the solution of $\theta'(z)$ from (20a), a similar approach to that in Section 4.1 can be employed and the result is

$$\theta'(z) = -\mathrm{i}h_0^*(z-\sqrt{z^2-a^2}), \tag{30a}$$

where

$$h_0^* = \frac{h_0(k_1+k_2)}{2k_1k_2}. \tag{30b}$$

With the aid of residue theory, the following line integrals which are useful for the calculation of $\psi'(z)$ can be obtained in a similar manner to those described in Section 4.1;

$$\begin{gathered}
\frac{1}{2\pi\mathrm{i}}\int_L \frac{1}{s-z}[\mathbf{X}_0^+(s)]^{-1}\mathbf{t}_0\,\mathrm{d}s = \left\langle\left\langle\frac{1}{\chi_\alpha(z)}-(z+2\mathrm{i}\varepsilon_\alpha a)\right\rangle\right\rangle\mathbf{t}_0^*, \\
\frac{1}{2\pi\mathrm{i}}\int_L \frac{s}{s-z}[\mathbf{X}_0^+(s)]^{-1}\mathbf{e}_k\,\mathrm{d}s = \left\langle\left\langle\frac{z}{\chi_\alpha(z)}-\left[z(z+2\mathrm{i}\varepsilon_\alpha a)-\frac{a^2}{2}(1+4\varepsilon_\alpha^2)\right]\right\rangle\right\rangle\mathbf{e}_k^*, \\
\frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathrm{i}\sqrt{a^2-s^2}}{s-z}[\mathbf{X}_0^+(s)]^{-1}\mathbf{e}_k\,\mathrm{d}s = \left\langle\left\langle\frac{\sqrt{z^2-a^2}}{\chi_\alpha(z)}-[z(z+2\mathrm{i}\varepsilon_\alpha a)-a^2(1+2\varepsilon_\alpha^2)]\right\rangle\right\rangle\mathbf{e}_k^*,
\end{gathered} \tag{31a}$$

where

$$\begin{gathered}
\mathbf{t}_0^* = \mathbf{\Lambda}^{-1}(\mathbf{I}+\bar{\mathbf{M}}^{*\;\;1}\mathbf{M}^*)^{-1}\mathbf{t}_0, \quad \mathbf{e}_k^* = \mathbf{\Lambda}^{-1}(\mathbf{I}+\bar{\mathbf{M}}^{*\;\;1}\mathbf{M}^*)^{-1}\mathbf{e}_k, \quad k = 1,2, \\
\chi_\alpha(z) = \frac{1}{\sqrt{z^2-a^2}}\left(\frac{z-a}{z+a}\right)^{\mathrm{i}\varepsilon_\alpha}.
\end{gathered} \tag{31b}$$

Applying the results of (30) and (31), the arbitrary polynomial $\mathbf{p}_n(z)$ in (20b) can now be determined by the infinity condition,

$$\psi'(z) \to 0, \quad \text{as } |z| \to \infty,$$

and the single-valuedness requirement,

$$\int_{-a}^{a}[\psi'(x_1^+)-\psi'(x_1^-)]\,\mathrm{d}x_1 = 0.$$

The result is

$$\mathbf{p}_n(z) = -\mathrm{i}h_0^*\left\{\frac{a^2}{4}\mathbf{e}_1^*+\langle\langle 4\mathrm{i}\varepsilon_\alpha az-a^2(\tfrac{1}{4}+4\varepsilon_\alpha^2)\rangle\rangle\mathbf{e}_2^*\right\}. \tag{32}$$

In the derivation of (32) the following integrals calculated by the residue theory have been used:

$$\int_{-a}^{a} \frac{1}{\sqrt{a^2-t^2}} \left(\frac{a-t}{a+t}\right)^{i\varepsilon_\alpha} \mathrm{d}t = \frac{\pi}{\cosh \pi\varepsilon_\alpha}, \quad \int_{-a}^{a} \frac{t}{\sqrt{a^2-t^2}} \left(\frac{a-t}{a+t}\right)^{i\varepsilon_\alpha} \mathrm{d}t = \frac{-2i\pi a\varepsilon_\alpha}{\cosh \pi\varepsilon_\alpha},$$

$$\int_{-a}^{a} \frac{t^2}{\sqrt{a^2-t^2}} \left(\frac{a-t}{a+t}\right)^{i\varepsilon_\alpha} \mathrm{d}t = \frac{\pi a^2}{2\cosh \pi\varepsilon_\alpha}.$$

Combining the results of (30)–(32), the final simplified solution for $\psi'(z)$ is found to be

$$\psi'(z) = -\Lambda\{\mathbf{J}_0(z)\mathbf{t}_0^* + ih_0^*(\mathbf{J}_1(z)\mathbf{e}_1^* + \mathbf{J}_2(z)\mathbf{e}_2^*)\}, \tag{33a}$$

where

$$\mathbf{J}_0(z) = \langle\langle 1-(z+2i\varepsilon_0 a)\chi_\alpha(z)\rangle\rangle, \quad \mathbf{J}_1(z) = \langle\langle z-\sqrt{z^2-a^2}-\tfrac{1}{4}a^2\chi_\alpha(z)\rangle\rangle,$$

$$\mathbf{J}_2(z) = \langle\langle z+\sqrt{z^2-a^2}-(2z^2-\tfrac{5}{4}a^2)\chi_\alpha(z)\rangle\rangle. \tag{33b}$$

The solutions for the isothermal interface crack problems are found by letting $h_0^* = 0$, which can be proved to be identical to those presented in Wu (1990) and Suo (1990a). The fracture parameters such as the stress intensity factors, crack opening displacements and the energy release rate can then be obtained from this solution and the proper definition for the interface crack problems (Wu, 1990; Suo, 1990a; Hwu, 1991).

5. CONCLUSIONS

A general solution for the thermoelastic collinear interface cracks between dissimilar anisotropic media has been obtained by applying the extended version of Stroh's formalism and a special technique of analytical continuation. The general solution is valid when the heat eigenvalue and the elasticity eigenvalues are distinct. For the case that they are repeated, a small perturbation of the material constants can be employed to avoid the degenerate problem. Otherwise, a modified solution should be developed. For the solutions in which the eigenvectors have been replaced by the fundamental matrices such as S, H, L and $\bar{\gamma}_1^*$, $\bar{\gamma}_2^*$ through the use of identities, the problems of repeated eigenvalues disappear, which has been demonstrated in the case of homogeneous media.

Acknowledgement—The author would like to thank the support by the National Science Council, Republic of China, through Grant No. NSC 80-0401-E-006-02.

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APPENDIX: SOLUTIONS TO THE HILBERT PROBLEM OF VECTOR FORM

The Hilbert problem is usually expressed in the form of scalar functions,

$$F^+(t)-gF^-(t) = f(t) \quad \text{on } L, \quad \text{except at the ends,} \tag{A1}$$

where g in general is a complex constant. The solution to this problem with $g \neq 1$ has been shown in Muskhelishvili (1954) as

$$F(z) = \frac{\chi_0(z)}{2\pi i}\int_L \frac{f(t)\,\mathrm{d}t}{\chi_0^+(t)(t-z)} + \chi_0(z)p_n(z) \tag{A2}$$

where

$$\chi_0(z) = \prod_{j=1}^{n} (z-a_j)^{-\gamma}(z-b_j)^{\gamma-1},$$

$\gamma = (1/2\pi i)\ln(g)$ and $p_n(z)$ is an arbitrary polynomial with the degree not higher than n. For $g = 1$, the solution is

$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)\,\mathrm{d}t}{t-z} + p_n(z). \tag{A3}$$

To find the solution for the vector form expression,

$$\boldsymbol{\psi}'(x_1^+) + \bar{\mathbf{M}}^{*-1}\mathbf{M}^*\boldsymbol{\psi}'(x_1^-) = \mathbf{t}, \quad x_1 \text{ on } L \text{ except at the ends,} \tag{A4}$$

a similar approach can be employed. Firstly, a solution will be studied which may have a pole of arbitrary order at infinity, and we begin with the homogeneous problem,

$$\boldsymbol{\psi}'(x_1^+) + \bar{\mathbf{M}}^{*-1}\mathbf{M}^*\boldsymbol{\psi}'(x_1^-) = \mathbf{0}, \quad x_1 \in L. \tag{A5}$$

A particular solution $\boldsymbol{\psi}'_0$ of the problem will be sought in the form

$$\boldsymbol{\psi}'_0(z) = \prod_{j=1}^{n} (z-a_j)^{-(1+\delta)}(z-b_j)^{\delta}\boldsymbol{\lambda}, \tag{A6}$$

where δ is a complex constant and $\boldsymbol{\lambda}$ is a complex constant vector. The function $\boldsymbol{\psi}'_0(z)$ is holomorphic in the entire plane cut along L, if a definite branch of this function is selected. It is readily verified by an investigation of the variation in the argument of $z-a_j$ or $z-b_j$, when z describes a closed path beginning at point x_1 of the arc a_jb_j and leading, without intersecting L, from the left side of a_jb_j around the end a_j to the right side of the arc or around the end b_j, that

$$\boldsymbol{\psi}'_0(x_1^+) = e^{2\pi i\delta}\boldsymbol{\psi}'_0(x_1^-). \tag{A7}$$

Hence, $\boldsymbol{\psi}'_0(z)$ will satisfy the boundary condition (A5), provided

$$(e^{2i\pi\delta}\mathbf{I}+\bar{\mathbf{M}}^{*-1}\mathbf{M}^{*})\boldsymbol{\lambda}=\mathbf{0},$$

or

$$(\mathbf{M}^{*}+e^{2i\pi\delta}\bar{\mathbf{M}}^{*})\boldsymbol{\lambda}=\mathbf{0}. \tag{A8}$$

The explicit solution for the eigenvalue δ has been given by Ting (1986) as

$$\delta_1=-\tfrac{1}{2}+i\varepsilon,\quad \delta_2=-\tfrac{1}{2}-i\varepsilon,\quad \delta_3=-\tfrac{1}{2}, \tag{A9}$$

where

$$\varepsilon=\frac{1}{2\pi}\ln\frac{1+\omega}{1-\omega},\quad \omega=[-\tfrac{1}{2}\,\mathrm{tr}\,(\mathbf{WD}^{-1})^2]^{1/2},$$

tr stands for the trace of the matrix. Thus, a particular solution $\boldsymbol{\psi}_0(z)$ of the homogeneous problem has been found: it is given by (A6) with δ and $\boldsymbol{\lambda}$ determined by (A8). Since there are three eigenvalues from (A8), a linear combination of these particular solutions will still be one of the particular solutions, i.e.

$$\boldsymbol{\psi}_0'(z)=\mathbf{X}_0(x)\mathbf{p}_0, \tag{A10}$$

where

$$\mathbf{X}_0(z)=\boldsymbol{\Lambda}\boldsymbol{\Gamma}(z), \tag{A11}$$

and

$$\boldsymbol{\Lambda}=[\boldsymbol{\lambda}_1\quad\boldsymbol{\lambda}_2\quad\boldsymbol{\lambda}_3],\qquad \boldsymbol{\Gamma}(z)=\left\langle\!\left\langle\prod_{j=1}^{n}(z-a_j)^{-(1+\delta_\alpha)}(z-b_j)^{\delta_\alpha}\right\rangle\!\right\rangle. \tag{A12}$$

$\mathbf{p}_0$ is a coefficient vector. This particular solution does not vanish anywhere in the finite part of the plane and it is unbounded like $|z-a_j|^{-1/2}$ and $|z-b_j|^{-1/2}$ near the ends a_j and b_j, respectively.

The most general solution of the homogeneous problem will now be found which has a pole at infinity. For this purpose it will be noted that $\boldsymbol{\psi}_0'(z)=\mathbf{X}_0(z)\mathbf{p}_0$, being a solution of the homogeneous problem, satisfies the condition

$$\mathbf{X}_0^{+}\mathbf{p}_0+\bar{\mathbf{M}}^{*-1}\mathbf{M}^{*}\mathbf{X}_0^{-}\mathbf{p}_0=\mathbf{0},\quad x_1\in L.$$

Hence

$$\bar{\mathbf{M}}^{*-1}\mathbf{M}^{*}=-\mathbf{X}_0^{+}[\mathbf{X}_0^{-}]^{-1}, \tag{A13}$$

where $\mathbf{X}_0^{\pm}$ is the simplified notation for $\mathbf{X}_0(x_1^{\pm})$. By applying (A13), eqn (A5) becomes

$$[\mathbf{X}_0^{+}]^{-1}\boldsymbol{\psi}'(x_1^{+})-[\mathbf{X}_0^{-}]^{-1}\boldsymbol{\psi}'(x_1^{-})=\mathbf{0},\quad x_1\in L,$$

or

$$\boldsymbol{\psi}_*(x_1^{+})-\boldsymbol{\psi}_*(x_1^{-})=\mathbf{0},\quad x_1\in L, \tag{A14}$$

where $\boldsymbol{\psi}_*(z)$ denotes the sectionally holomorphic function $[\mathbf{X}_0(z)]^{-1}\boldsymbol{\psi}'(z)$. It follows from (A14) that $\boldsymbol{\psi}_*(z)$ is holomorphic in the entire plane, except at the point $z=\infty$, provided it is given suitable values on L. Further, since $\boldsymbol{\psi}_*(z)$ can only have a pole at infinity, it must, by the generalized Liouville theorem, be a polynomial. Thus, the most general solution of the homogeneous problem is given by

$$\boldsymbol{\psi}'^{(h)}(z)=\mathbf{X}_0(z)\mathbf{p}_n(z), \tag{A15}$$

where $\mathbf{p}_n(z)$ is an arbitrary polynomial vector. If it is desired to obtain a solution which is also holomorphic at infinity, it must be assumed that the degree of the polynomial $\mathbf{p}_n(z)$ does not exceed n. This follows from the behavior of $\mathbf{X}_0(z)$ at infinity as given in (A11).

Next consider the non-homogeneous problem. Using (A13), the boundary condition (A4) may be written as

$$[\mathbf{X}_0^{+}]^{-1}\boldsymbol{\psi}'(x_1^{+})-[\mathbf{X}_0^{-}]^{-1}\boldsymbol{\psi}'(x_1^{-})=[\mathbf{X}_0^{+}]^{-1}\mathbf{t},\quad x_1\in L,$$

or

$$\boldsymbol{\psi}_*(x_1^{+})-\boldsymbol{\psi}_*(x_1^{-})=[\mathbf{X}_0^{+}]^{-1}\mathbf{t},\quad x_1\in L, \tag{A16}$$

where $\boldsymbol{\psi}_*(z)=[\mathbf{X}_0(z)]^{-1}\boldsymbol{\psi}'(z)$. Each component of eqn (A16) is in the form of (A1) with $g=1$, hence, by (A3) we have

$$\boldsymbol{\psi}'(z)=\frac{1}{2\pi i}\mathbf{X}_0(z)\int_L\frac{1}{s-z}[\mathbf{X}_0^{+}(s)]^{-1}\mathbf{t}(s)\,\mathrm{d}s+\mathbf{X}_0(z)\mathbf{p}_n(z) \tag{A17}$$

where $\mathbf{p}_n(z)$ is an arbitrary polynomial vector with the degree not higher than n.